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Global Behavior of Limit Cycles in Rotated Vector Fields*

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We give a general definition for rotated vector fields and establish certain new theorems for the global behavior of limit cycles in the family of rotated vector fields which generalize, improve, or correct some known results. © 1999 Academic Press

The theory of rotated vector fields was originated by Duff [5] in 1953 and was extended by Chen [1–3] and Perko [8, 9]. This theory has proved to be immensely useful in research on limit cycles (see [1–3, 5, 7–11, 14–16]). In this section, we first introduce definitions of rotated vector fields given by Duff, Chen, and Perko, and then give a more general definition.

Consider a one-parameter family of vector fields of the form

$$\frac{dx}{dt} = f(x, \lambda), \tag{1.1}$$

where $x \in R^2$, $\lambda \in R$. Suppose the function f is of class C^1 on $R^2 \times R$. Then (1.1) defines a C^1 vector field $F(\lambda) = f(\cdot, \lambda)$.

Definition 1.1 (Duff [5]). Equation (1.1) constitutes a complete family, and the corresponding vector fields $F(\lambda)$ constitute a complete family of rotated vector fields for $0 \leqslant \lambda < \pi$ if

(a) The critical points of (1.1) are isolated and remain fixed as λ varies.

(b) At all ordinary points $f(x, \lambda) \wedge f_\lambda(x, \lambda) \neq 0$ where $a \wedge b = a_1 b_2 - a_2 b_1$.

(c) $f(x, \lambda + \pi) = -f(x, \lambda)$.

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Evidently condition (c) is a big restriction. To weaken it, Chen gave the following:

DEFINITION 1.2 (Chen [1, 2]). $F(\lambda)$ is said to constitute a family of rotated vector fields for $\lambda \in [0, T]$ if:

(a) The critical points of (1.1) are isolated and remain fixed as λ varies.

(b) At all ordinary points $f(x, \lambda) \wedge f_\lambda(x, \lambda) \geqslant 0$ (or $\leqslant 0$) and $\not\equiv 0$ along any nontrivial closed curve.

(c) For any $0 < \lambda_1 < \lambda_2 < T$ and $x \in R^2$

$$0 \leqslant \int_{\lambda_1}^{\lambda_2} \theta_\lambda(x, \lambda)\, d\lambda \leqslant \pi \qquad \left(\text{or } -\pi \leqslant \int_{\lambda_1}^{\lambda_2} \theta_\lambda(x, \lambda)\, d\lambda \leqslant 0\right),$$

where $\theta(x, \lambda)$ denotes the angle of $f(x, \lambda)$ with the positive x_1-axis.

Chen [3] also gave another definition, and it was refined in [16] as follows.

DEFINITION 1.3 ([3, 16]). $F(\lambda)$ is said to constitute a family of rotated vector fields for $\lambda \in (a, b)$ if:

(a) The critical points of (1.1) are isolated and remain fixed for $\lambda \in (a, b)$.

(b) For any $a < \lambda_1 < \lambda_2 < b$, $f(x, \lambda_1) \wedge f(x, \lambda_2) \geqslant 0$ (or $\leqslant 0$) and $\not\equiv 0$ along any periodic orbit of $F(\lambda_1)$ and $F(\lambda_2)$.

In 1975, Perko [8] introduced the definition of a semicomplete family of rotated vector fields as follows.

DEFINITION 1.4 (Perko [8]). $F(\lambda)$ is called a semicomplete family of rotated vector fields if:

(a) $f(x, \lambda)$ is analytic and the critical points of (1.1) remain fixed for $\lambda \in R$.

(b) At all ordinary points $f(x, \lambda) \wedge f_\lambda(x, \lambda) > 0$ for $\lambda \in R$.

(c) $\tan \theta(x, \lambda) \to \pm\infty$ as $\lambda \to \pm\infty$.

From [1–3, 5, 8, 9, 15, 16] we know that the following two conclusions hold if the conditions of one of Definitions 1.1–1.4 are satisfied.

(i) For $\lambda_1, \lambda_2 \in I$, $\lambda_1 \neq \lambda_2$, the periodic orbits of $F(\lambda_1)$ and $F(\lambda_2)$ do not intersect, where $I = [0, \pi]$, $[0, T]$, (a, b), or $(-\infty, +\infty)$ in the case that the conditions of Definition 1.1, 1.2, 1.3, or 1.4 hold.

(ii) Let $F(\lambda_0)$ have a limit cycle $L(\lambda_0)$, where λ_0 is an interior point of I. Then for $|\lambda-\lambda_0|$ sufficiently small, (1.1) has at least one limit cycle near $L(\lambda_0)$ if $L(\lambda_0)$ is stable or completely unstable, and has at least two limit cycles for λ varying in a suitable sense, and no periodic orbits for λ varying in the opposite sense if $L(\lambda_0)$ is semistable.

Recently, Perko [9] introduced the following definition.

DEFINITION 1.5 (Perko [9]). The system (1.1) defines a family of rotated vector fields (mod $G=0$) if the function $f(x, \lambda)$ is analytic and the critical points of (1.1) remain fixed, and if $f(x, \lambda) \wedge f_\lambda(x, \lambda) > 0$ at all regular points of (1.1) except those on the set of curves defined by $G(x, y)=0$, where G is an analytic function independent of λ.

Now we give our definition as follows.

DEFINITION 1.6. Suppose that the function $f(x, \lambda)$ is analytic on $D\times I$, where D is a connected set of R^2 and I is an interval. If the critical points of (1.1) located in the interior of D remain fixed, and if for any interior point $(x_0, \lambda_0)\in D\times I$ with x_0 a regular point there exists a neighborhood $D_0\subset D$ of x_0 and $\varepsilon>0$ with $\lambda_0+\varepsilon\in I$ such that

$$f(x, \lambda_0) \wedge f(x, \lambda) \geqslant 0 \qquad (\text{or } \leqslant 0) \qquad \text{for} \quad x\in D_0, \qquad \lambda\in(\lambda_0, \lambda_0+\varepsilon), \tag{1.2}$$

and

$$f(x, \lambda_0) \wedge f(x, \lambda) \not\equiv 0 \qquad \text{for} \quad \lambda\in(\lambda_0, \lambda_0+\varepsilon) \tag{1.3}$$

along any nontrivial invariant closed curve in D of the vector field $F(\lambda_0)$, we say that the system (1.1) defines a family of rotated vector fields with $(x, \lambda)\in D\times I$.

Perko [9] studied the limit cycle bifurcations for the type of "mod $G=0$" system. Most results required that $G(x, y)\not\equiv 0$ along any closed orbits of (1.1). This condition is not given explicitly in Definition 1.5. Hence, to show that Definition 1.6 is very general, we prove

PROPOSITION 1.1. (i) *We have that* (1.2) *holds if and only if*

$$f(x, \lambda) \wedge f_\lambda(x, \lambda) \geqslant 0 \qquad (\text{or } \leqslant 0) \qquad \text{for} \quad (x, \lambda)\in D\times I. \tag{1.4}$$

(ii) *Let* (1.2) *hold. If*

$$f(x, \lambda) \wedge f_\lambda(x, \lambda) \not\equiv 0 \qquad \text{for} \quad \lambda\in I \tag{1.5}$$

along any nontrivial invariant closed curve in D of the vector field $F(\lambda)$, then (1.3) *holds. However, the converse is not true.*

Proof. (i) Suppose (1.2) holds. We have $f(x_0, \lambda_0) \wedge [f(x_0, \lambda) - f(x_0, \lambda_0)] \geqslant 0$ (or $\leqslant 0$) for $0 < \lambda - \lambda_0 < \varepsilon$. This gives that $f(x_0, \lambda_0) \wedge f_\lambda(x_0, \lambda_0) \geqslant 0$ (or $\leqslant 0$), and (1.4) follows. Let (1.4) hold. From [5] we have

$$\theta_\lambda(x, \lambda) \equiv \frac{1}{|f(x, \lambda)|} f(x, \lambda) \wedge f_\lambda(x, \lambda) \geqslant 0 \qquad (\text{or } \leqslant 0) \tag{1.6}$$

at all regular points. For definiteness, we suppose $\theta_\lambda(x, \lambda) \geqslant 0$. Let (x_0, λ_0) be an interior point of $D \times I$ with x_0 a regular point. Let $\varepsilon^* > 0$ be such that $\lambda_0 + \varepsilon^* \in I$. If $\theta(x_0, \lambda) - \theta(x_0, \lambda_0) < \pi/2$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon^*]$, then $\theta(x, \lambda) - \theta(x, \lambda_0) < \pi/2$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon^*]$ and x near x_0. If $\theta(x_0, \lambda) - \theta(x_0, \lambda_0) = \pi/2$ for some $\lambda \in [\lambda_0, \lambda_0 + \varepsilon^*]$ then for x near x_0 there exists a continuous function $\lambda = \lambda(x) \in (\lambda_0, \lambda_0 + \varepsilon^*)$ such that $\theta(x, \lambda) - \theta(x, \lambda_0) < \pi/4$ $(= \pi/4)$ for $\lambda_0 \leqslant \lambda < \lambda(x) (\lambda = \lambda(x))$. Hence, noting (1.6), there exists a neighborhood D_0 of x_0 and $\varepsilon > 0$ with $\lambda_0 + \varepsilon \in I$ such that

$$0 \leqslant \theta(x, \lambda) - \theta(x, \lambda_0) < \frac{\pi}{2} \qquad \text{for} \quad x \in D_0, \qquad \lambda \in (\lambda_0, \lambda_0 + \varepsilon). \tag{1.7}$$

Note that $\tan \theta(x, \lambda) = f_2(x, \lambda)/f_1(x, \lambda)$. We have

$$\tan(\theta(x, \lambda) - \theta(x, \lambda_0)) = \frac{\tan \theta(x, \lambda) - \tan \theta(x, \lambda_0)}{1 + \tan \theta(x, \lambda) \tan \theta(x, \lambda_0)}$$

$$= \frac{f(x, \lambda_0) \wedge f(x, \lambda)}{|f(x, \lambda_0)| \cdot |f(x, \lambda)| \cos(\theta(x, \lambda) - \theta(x, \lambda_0))}.$$

It follows from (1.7) that $f(x, \lambda_0) \wedge f(x, \lambda) \geqslant 0$ for $x \in D_0$, $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

(ii) Suppose that (1.3) does not hold. Then there exist a nontrivial invariant closed curve $L(\lambda_0)$ of $F(\lambda_0)$ and a sequence $\lambda_n > \lambda_0$ with $\lambda_n \to 0$ such that

$$f(x, \lambda_0) \wedge f(x, \lambda_n) = 0 \qquad \text{for} \quad x \in L(\lambda_0).$$

It follows that $f(x, \lambda_0) \wedge f_\lambda(x, \lambda_0) = 0$ for $x \in L(\lambda_0)$. Thus, we have proved that (1.5) implies (1.3). To prove that the converse is not true, consider the family of vector fields $F(\lambda)$ given by

$$f(x, \lambda) = (x_2 + x_1(x_1^2 + x_2^2 - \lambda)^3, \, -x_1)^{\mathrm{T}}, \tag{1.8}$$

where $x=(x_1, x_2)$, $\lambda \in I=(0, +\infty)$. It is direct that

$$f(x, \lambda) \wedge f_\lambda(x, \lambda) = -3x_1^2(x_1^2+x_2^2-\lambda)^2,$$
$$f(x, \lambda_1) \wedge f(x, \lambda_2) = x_1^2(\lambda_1-\lambda_2)\, g(x_1^2+x_2^2, \lambda_1, \lambda_2),$$

where $g(u, \lambda_1, \lambda_2)=3u^2-3(\lambda_1+\lambda_2)\,u+\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2>0$ for $0<\lambda_1<\lambda_2$ and $u \geqslant 0$. Note that the circle $x_1^2+x_2^2=\lambda$ is the only periodic orbit of the vector field $F(\lambda)$. We see that (1.2) and (1.3) hold for the chosen family of vector fields, but (1.5) is not satisfied. The proof is completed.

From the above proposition, we see that Definition 1.6 is more general than Definitions 1.1–1.5 but less general than a definition of Chen [3] where only the condition (1.2) was required. In the next section we will prove that the conditions in Definition 1.6 keep most nice properties of rotated vector fields. We will also illustrate that condition (1.2) alone is not enough to keep some well-known properties of rotated vector fields.

2. GLOBAL BEHAVIOR OF LIMIT CYCLES

In the section we establish a general theory on the global behavior of limit cycles in rotated vector fields as the parameter varies. When we discuss rotated vector fields we always mean that the conditions of Definition 1.6 are satisfied.

First, we have the following fundamental result:

THEOREM 2.1 (Nonintersection Theorem). *Suppose that the system* (1.1) *defines a family of rotated vector fields with* $(x, \lambda) \in D \times I$. *Let for* $\lambda_1, \lambda_2 \in I$, $\lambda_1 \neq \lambda_2$, *the vector fields* $F(\lambda_1)$ *and* $F(\lambda_2)$ *have periodic orbits* L_1 *and* L_2 *in* D, *respectively. Then either* (i) $L_1=L_2$ *or* (ii) $L_1 \cap L_2=\phi$ *if* L_1 *and* L_2 *have the same orientation and surround the same critical points of* (1.1). *Moreover, the case* (i) *cannot occur if* $0<|\lambda_1-\lambda_2| \ll 1$.

Proof. Suppose that L_1 and L_2 have the same orientation and surround the same critical points. Without loss of generality, we assume that L_1 and L_2 are oriented clockwise. if the conclusion is not true, then $L_1 \neq L_2$ and $L_1 \cap L_2 \neq \phi$. Noting that ($f(x, \lambda)$ is analytic and L_1, L_2 have the same orientation and surround the same critical points, there are two cases to consider (see Fig. 1):

Case (i). L_1 and L_2 are tangent at a point $A \in L_1 \cap L_2$, and either (a) $L_2-\{A\} \subset \operatorname{Int} L_1$ or (b) $L_1-\{A\} \subset \operatorname{Int} L_2$.

Case (ii). There exists an arc $l_2 \subset L_2$ with endpoints $A_1, A_2 \in L_2$ such that $l_2-\{A_1, A_2\} \subset \operatorname{Int} L_1$, $A_1, A_2 \in L_1 \cap l_2$.

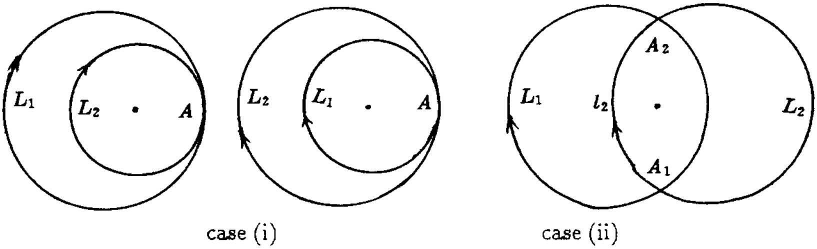

FIG. 1.

We first consider Case (i)(a). For definiteness, from (1.2) and (1.6) we can suppose that

$$\lambda_1 < \lambda_2, \; \theta_\lambda(x, \lambda) = \frac{1}{|f(x, \lambda)|} f(x, \lambda) \wedge f_\lambda(x, \lambda) \geqslant 0. \tag{2.1}$$

Let x_0 denote the coordinate of A. Then we have

$$\theta(x_0, \lambda_1) = \theta(x_0, \lambda_2) \equiv \theta_0. \tag{2.2}$$

If $\theta_0 \neq 0 \pmod{2\pi}$, we may suppose

$$\theta(x, \lambda_i) \in (0, 2\pi) \qquad \text{for} \quad |x - x_0| \ll 1, \qquad i = 1, 2. \tag{2.3}$$

If $\theta_0 = 0 \pmod{2\pi}$, we may suppose

$$\theta(x, \lambda_i) \in (-\pi, \pi) \qquad \text{for} \quad |x - x_0| \ll 1, \qquad i = 1, 2. \tag{2.4}$$

Then from (2.1), (2.2), and (2.3) or (2.4), we have

$$0 \leqslant \theta(x, \lambda_2) - \theta(x, \lambda_1) < 2\pi \qquad \text{for} \quad |x - x_0| \ll 1. \tag{2.5}$$

This implies that the positive orbit of $F(\lambda_2)$ passing through any point of L_1 near A must cross L_1 from its interior into its exterior. Therefore, there is a point $B \in L_2$ near A with $B \neq A$ such that B is outside L_1. This contradicts the fact that $L_2 - \{A\} \subset \operatorname{Int} L_1$. Hence, Case (i)(a) cannot occur. In the same way, Case (i)(b) cannot occur either.

Let us consider Case (ii). From the continuity theorem for solutions, there must exist a point $x_0 \in L_1 \cap \operatorname{Ext} L_2$ (here, $\operatorname{Ext} L_2$ means the exterior of L_2) such that the field vector $f(x_0, \lambda_2)$ is tangent to L_1 at x_0. In other words, we have

$$\theta(x_0, \lambda_2) = \theta(x_0, \lambda_1) \qquad \text{or} \qquad \theta(x_0, \lambda_2) - \theta(x_0, \lambda_1) = \pi,$$

where we have used (2.1). In the former case, we have that (2.5) holds, which yields a contradiction as in Case (i)(a). Therefore, we must have

$$\theta(x_0, \lambda_2) - \theta(x_0, \lambda_1) = \pi. \tag{2.6}$$

Let M (or N) denote the number of inner (or outer) tangent points which are located on $L_1 \cap \operatorname{Ext} L_2$ and at which the vector $f(x, \lambda_2)$ is tangent to L_1. Since f is analytic, the numbers M and N are finite. Furthermore, from (2.6) and Fig. 1(ii), it is easy to see that M and N satisfy $M = N + 1$ (see Fig. 2).

Let $l_3 = \operatorname{Arc} B_1 B_2 \subset \operatorname{Ext} L_2$ with $B_1, B_2 \in L_1$ be an orbit segment of the vector field $F(\lambda_2)$ near $l_2 = \operatorname{Arc} A_1 A_2 \subset \operatorname{Int} L_1$. The points B_1 and B_2 divide L_1 into two parts. One of them is in the exterior of L_2, denoted by l_1. When l_3 and l_2 are close enough, the $M + N$ inner and outer tangent points are all on l_1. Following Ye [14, Chapt. 1], we can construct a segment l_2^* between l_2 and l_3 such that (a) l_1 and l_2^* form a smooth closed curve, denoted by L^*, and that (b) there is a unique tangent point of $F(\lambda_2)$ to l_2^* on l_2^* which is an outer tangent point with respect to $L^* = l_1 \cup l_2^*$. It follows that the vector field $F(\lambda_2)$ has exactly M inner tangent points and $N+1$ outer tangent points on the closed curve L^*. Thus, by the Poincaré index theorem [13, 14] the sum of indices of critical points of $F(\lambda_2)$ in the interior of L^* is $1 + \frac{1}{2}(\mathrm{M} - \mathrm{N} - 1) = 1$. This yields that the curve L^* surrounds at least one critical point of (1.1), contradicting the fact that L_1 and L_2 surround the same critical points.

Now suppose $L_1 = L_2$ for some λ_1, λ_2 with $0 < |\lambda_1 - \lambda_2| \ll 1$. Then for any $x \in L_1$, the field vectors $f(x, \lambda_1)$ and $(f(x, \lambda_2)$ are always parallel. This gives that $f(x, \lambda_1) \wedge f(x, \lambda_2) = 0$ for all $x \in L_1$, contradicting condition (1.3). This ends the proof.

Recall that a singular closed orbit is a closed curve which consists of a finite number of critical points and orbits connecting them, on which the field vectors always have the same sense. From the proof of Theorem 1.1

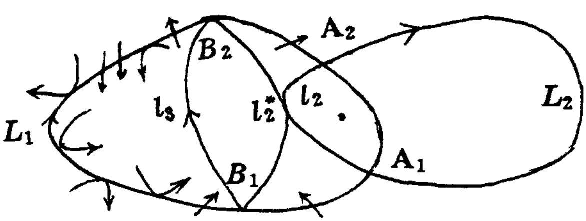

FIG. 2. ($M = 3$, $N = 2$)

and a generalization of the Poincaré index theorem of Ye [14], Theorem 2.1 remains true if L_1 or L_2 is a singular closed orbit.

We remark that Theorem 2.1 does not hold if L_1 and L_2 have different orientations. This can be illustrated by the system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y, \lambda) \cos \lambda + Q(x, y, \lambda) \sin \lambda \equiv P^*(x, y, \lambda), \\ \frac{dy}{dt} &= -P(x, y, \lambda) \sin \lambda + Q(x, y, \lambda) \cos \lambda \equiv Q^*(x, y, \lambda), \end{aligned} \tag{2.7}$$

where $\lambda \in R$ and

$$\begin{aligned} &P = H_y + H_x H, \qquad Q = -H_x + H_y H, \\ &H = (1 + \tfrac{1}{8} \sin \lambda)\, x^2 + (1 + \tfrac{1}{8} \sin \lambda)\, y^2. \end{aligned} \tag{2.8}$$

it is easy to see that

$$\begin{vmatrix} P^* & Q^* \\ P^*_\lambda & Q^*_\lambda \end{vmatrix} = -[P^2 + Q^2 + P_\lambda Q - P Q_\lambda]. \tag{2.9}$$

From (2.7) we have

$$\begin{aligned} &P^2 + Q^2 = (H_x^2 + H_y^2)(1 + H^2), \qquad H_x^2 + H_y^2 \geqslant \tfrac{49}{16}(x^2 + y^2), \\ &P_\lambda Q - P Q_\lambda = (H_y H_{x\lambda} - H_x H_{y\lambda})(1 + H^2) - H_\lambda (H_x^2 + H_y^2). \end{aligned} \tag{2.10}$$

Hence, we have

$$\begin{aligned} \left| \frac{P_\lambda Q - P Q_\lambda}{P^2 + Q^2} \right| &= \left| \frac{H_y H_{x\lambda} - H_x H_{y\lambda}}{H_x^2 + H_y^2} - \frac{H_\lambda}{1 + H^2} \right| \\ &= \left| \frac{-xy \cos \lambda}{H_x^2 + H_y^2} - \frac{y^2 - x^2}{8(1 + H^2)} \right| \\ &\leqslant \frac{16xy}{49(x^2 + y^2)} + \frac{x^2 + y^2}{8[1 + (7(x^2 + y^2)/8)^2]} \\ &\leqslant \frac{8}{49} + \frac{1}{14} < 1. \end{aligned}$$

Therefore from (2.8) we have

$$\begin{vmatrix} P^* & Q^* \\ P^*_\lambda & Q^*_\lambda \end{vmatrix} < 0 \qquad \text{for} \quad (x, y) \in R^2, \quad \lambda \in R.$$

According to Definition 1.5 or 1.6 and Proposition 1.1, the system (2.6) defines a family of rotated vector fields with $(x, y, \lambda) \in R^2 \times R$. Note that

$$\frac{dH}{dt} = (H_x^2 + H_y^2) \cos \lambda (H - \tan \lambda) \tag{2.11}$$

along orbits of (2.6). The closed curve $L(\lambda)$: $H = \tan \lambda$ is a limit cycle of (2.6) for $\lambda \in R - \{k\pi + \pi/2 \mid k = 0, 1, ...\}$ with $\tan \lambda > 0$. Choose $\lambda_1 = \pi/4$, $\lambda_2 = 5\pi/4$. Then $L(\lambda_1)$ and $L(\lambda_2)$ are elliptic curves given by

$$\left(1 - \frac{\sqrt{2}}{16}\right) x^2 + \left(1 + \frac{\sqrt{2}}{16}\right) y^2 = 1$$

and

$$\left(1 + \frac{\sqrt{2}}{16}\right) x^2 + \left(1 - \frac{\sqrt{2}}{16}\right) y^2 = 1$$

respectively. The two curves surround the origin which is the only critical point of (2.6) and have opposite orientations. However, they have exactly four intersection points.

This example shows that in Theorem 2.1 the condition that L_1 and L_2 have the same orientation is necessary. This condition or the requirement that the system defines a semicomplete family of rotated vector fields should be added to the important nonintersection theorem given by Perko in [9, 11].

Next, we prove

THEOREM 2.2. *Suppose* (1.1) *defines a family of rotated vector fields with* $(x, \lambda) \in D \times I$. *Assume that for* $\lambda_0 \in \operatorname{Int} I$, (1.1) *has a periodic orbit* $L_0 \subset \operatorname{Int} D$. *Then for each* $\lambda \in I$ *near* λ_0 *the following hold*:

(i) *If* L_0 *is stable or completely unstable*, (1.1) *has a unique limit cycle* $L(\lambda)$ *which expands or contracts monotonically as* λ *varies in a fixed sense.*

(ii) *If* L_0 *is semistable, then it splits into a stable and an unstable limit cycle as* λ *varies in a suitable sense*; L_0 *disappears as* λ *varies in the opposite sense.*

(iii) *If* L_0 *is nonisolated, then* (1.1) *has no periodic orbits in a neighborhood of* L_0 *for all* λ *near* λ_0 *with* $\lambda \neq \lambda_0$.

Proof. If the periodic orbit L_0 is nonisolated (i.e., every neighborhood of L_0 contains a periodic orbit), from Theorem 2.1 we know that (1.1) has no periodic orbits near L_0 for $0 < |\lambda - \lambda_0| \ll 1$. Thus, we suppose L_0 is isolated. Let $x = u(t)$ $(0 \leqslant t \leqslant T_0)$ be a parameter representation of L_0 with T_0 the least period of L_0. Following [4] to introduce a coordinate transformation of the form $x = u(\theta) + Z(\theta)\, p$, where $Z(\theta) = (-V_2(\theta), V_1(\theta))^{\mathrm{T}}$, $(V_1(\theta), V_2(\theta))^{\mathrm{T}} = u'(\theta)/|u'(\theta)|$, $0 \leqslant \theta \leqslant T_0$, we can get an analytic T_0-periodic system from (1.1):

$$\frac{dp}{d\theta} = R(\theta, p, \lambda). \tag{2.12}$$

Let $G(r, \lambda)$ denote the displacement function of the periodic system. Obviously, there are a natural number k and a constant $g_k \neq 0$ such that $G(r, \lambda_0) = g_k r^k + O(r^{k+1})$ which says that L_0 is a limit cycle of multiplicity k. If k is odd, $G(r, \lambda)$ has at least one root $r(\lambda)$ for all λ near λ_0 with $r(\lambda_0) = 0$. By Theorem 2.1 or (1.2) and (1.3), $r(\lambda) \not\equiv 0$. Hence, by the Newton polygon method [4] we have $r(\lambda) = \beta(\lambda - \lambda_0)^{\alpha}\,(1 + o(1))$ for a constant $\beta \neq 0$ and a rational number $\alpha > 0$. From (1.2) and (1.3) it is easy to see that $G(0, \lambda)(\lambda - \lambda_0)$ has a fixed sign for $0 < |\lambda - \lambda_0| \ll 1$. It follows that $(\lambda - \lambda_0)\, r(\lambda)$ has a fixed sign. Hence, the limit cycle $L(\lambda)$ corresponding to $r(\lambda)$ covers a neighborhood of L_0 as λ varies near λ_0. Then by Theorem 2.1 again $L(\lambda)$ is the only limit cycle of $F(\lambda)$ for $|\lambda - \lambda_0|$ small.

Let k be even. For definiteness, assume that

$$g_k > 0 \qquad \text{and} \qquad G(0, \lambda)(\lambda - \lambda_0) < 0 \qquad \text{for} \quad 0 < |\lambda - \lambda_0| \ll 1.$$

Then for $0 < \lambda - \lambda_0 \ll 1$, $G(0, \lambda) < 0$. Since $G(r, \lambda_0) = g_k r^k + O(r^{k+1}) > 0$ for $0 < |r| \ll 1$, we have that $G(r, \lambda)$ has a positive root $r_1(\lambda)$ and a negative root $r_2(\lambda)$ for $0 < \lambda - \lambda_0 \ll 1$. Let $L_i(\lambda)$ be the limit cycles corresponding to $r_i(\lambda)$, $i = 1, 2$. Then, as before, $L_1(\lambda)$ and $L_2(\lambda)$ together cover a neighborhood of L_0 for $0 \leqslant \lambda - \lambda_0 \ll 1$. It follows from Theorem 2.1 that $L_1(\lambda)$ and $L_2(\lambda)$ are the only limit cycles of $F(\lambda)$ for $0 < \lambda - \lambda_0 \ll 1$ and that $F(\lambda)$ has no periodic orbits for $0 < \lambda_0 - \lambda \ll 1$. The proof is completed.

If $f(x, \lambda_0) \wedge f_\lambda(x, \lambda_0) \not\equiv 0$ along L_0, then from Proposition 1.1 we have

$$\int_{L_0} \exp\left(-\int_0^t \operatorname{tr} f_x(x, \lambda_0)\, ds\right) f(x, \lambda_0) \wedge f_\lambda(x, \lambda_0)\, dt \neq 0.$$

in this case Theorem 2.2 can be implied from Theorem 7.2 [7]. The first two conclusions can be deduced by a theorem in the appendix of [10]. However, if $f(x, \lambda_0) \wedge f_\lambda(x, \lambda_0) \equiv 0$ along L_0, Theorem 2.2 no longer holds. To see this, consider the analytic system

$$\frac{dx}{dt} = y + x(x^2 + y^2 - 1)^2 [(x^2 + y^2 - 1)^k - \lambda],$$

$$\frac{dy}{dt} = -x,$$

where $\lambda > -1$, $k = 1$ or 2. For (2.11), we have

$$f(x, y, \lambda_0) \wedge f(x, y, \lambda) = x^2(x^2 + y^2 - 1)^2 (\lambda_0 - \lambda).$$

Thus, for (2.11) the condition (1.3) is not satisfied on any connected region D containing the circle L_0: $x^2 + y^2 = 1$. When $\lambda = 0$, (2.11) has a unique limit cycle of multiplicity $k+2$. If $k = 1$, then for $0 < \lambda \ll 1$ (2.11) has always a semistable limit cycle L_0 and a hyperbolic limit cycle $L(\lambda)$: $x^2 + y^2 = \lambda + 1$. If $k = 2$, then for $0 < \lambda \ll 1$ (2.11) always has a semistable limit cycle L_0 and two hyperbolic limit cycles $L_j(\lambda)$: $x^2 + y^2 = 1 + (-1)^j \sqrt{\lambda}$, $j = 1, 2$. As λ varies from zero to negative, the limit cycles $L_1(\lambda)$ and $L_2(\lambda)$ appear, and the limit cycle L_0 remains semistable.

Further, we study Hopf bifurcations for (1.1). Suppose that (1.1) has a nonhyperbolic elementary critical point for some $\lambda = \lambda_0 \in I$. We can assume that the critical point is at the origin for $|\lambda - \lambda_0|$ small. That is to say, $f(0, \lambda) = 0$ for $|\lambda - \lambda_0|$ small.

THEOREM 2.3. *Suppose* (1.1) *defines a family of rotated vector fields. Then we have the following*:

(i) *If the origin is a center point of* $F(\lambda_0)$, *then* (1.1) *has no periodic orbits near the origin for all* $\lambda \in I$ *near* λ_0 *with* $\lambda \neq \lambda_0$.

(ii) *If the origin is a focus of* $F(\lambda)$ *and is stable for* $\lambda \leqslant \lambda_0$ (*or* $\lambda \geqslant \lambda_0$) *and unstable for* $\lambda > \lambda_0$ (*or* $\lambda < \lambda_0$), *then* (1.1) *has a unique limit cycle for* $\lambda > \lambda_0$ (*or* $\lambda < \lambda_0$) *and no periodic orbits for* $\lambda \leqslant \lambda_0$ (*or* $\lambda \geqslant \lambda_0$).

Proof. Conclusion (i) is a direct corollary of Theorem 2.1. For conclusion (ii), let the origin be weak focus of $F(\lambda_0)$. By introducing the polar coordinate change to (1.1) we can obtain an analytic 2π-periodic system of the form (2.10). We also use $G(r, \lambda)$ to denote the displacement function of the obtained periodic system. Then for $|r| \ll 1$ we have

$$G(r, \lambda_0) = a_{2k} r^{2k+1} + O(r^{2k+2}), \qquad a_{2k} \neq 0, \qquad k \geqslant 1. \tag{2.13}$$

Note that $G(0, \lambda) = 0$ for $|\lambda - \lambda_0|$ small. The conclusion (ii) can be verified in a similar way to Theorem 2.2. The proof is complete.

We note that if the function f in (1.1) is only of class C^∞ then by using the Malgrange Preparation Theorem [4] one can show that Theorem 2.3 remains true as long as (2.12) holds. However, the theorem is no longer valid if (2.12) fails. To see this, let us consider the C^∞ system

$$\begin{aligned} \frac{dx}{dt} &= -y + x \tan h(r) \equiv P_0(x, y), \\ \frac{dy}{dt} &= x + y \tan h(r) \equiv Q_0(x, y), \end{aligned} \tag{2.14}$$

where $r = \sqrt{x^2 + y^2}$ and

$$h(r) = \begin{cases} \frac{1}{3} e^{-1/r}(\sin 1/r + \frac{3}{4}\sqrt{2}), & r > 0, \\ 0, & r = 0. \end{cases} \tag{2.15}$$

Choose $D = \{x, y) \mid 0 \leqslant r < 1\}$ and embed (2.13) into the family of uniform rotations

$$\begin{aligned} \frac{dx}{dt} &= P_0(x, y) \cos \lambda - Q_0(x, y) \sin \lambda \equiv P(x, y, \lambda), \\ \frac{dy}{dt} &= P_0(x, y) \sin \lambda + Q_0(x, y) \cos \lambda \equiv Q(x, y, \lambda), \end{aligned} \tag{2.16}$$

which satisfies $\theta_\lambda \equiv 1$.

The functions P and Q are of class C^∞ on $D \times R$. It is direct that along the orbits of (2.15) we have

$$\frac{dr^2}{dt} = 2r^2 \cos \lambda [\tan h\,(r) - \tan \lambda].$$

Note that $h(r) > 0$ for $r > 0$ and $h(0+0) = 0$. This implies that for $|\lambda|$ small the origin is unstable (stable) for $\lambda \leqslant 0$ (> 0) and that (2.15) has a periodic orbit near the origin of and only if the equation

$$h(r) = \lambda \tag{2.17}$$

has a positive solution with respect to r for $\lambda > 0$. We have

$$\frac{dh}{dr}(r) = \frac{1}{3} r^{-2} e^{-(1/r)} \left(\sin \frac{1}{r} - \cos \frac{1}{r} + \frac{3}{4}\sqrt{2} \right)$$

and

$$\frac{dh}{dr}\left(\left(2k\pi - \frac{\pi}{4}\right)^{-1}\right) < 0 < \frac{dh}{dr}\left(\left(2k\pi + \frac{3}{4}\pi\right)^{-1}\right), \qquad k \geqslant 1.$$

Hence we see that there exists a sequence $\{\lambda_k\} \subset (0, 1)$ with $\lambda_k \to 0$ as $k \to +\infty$ such that the equation (2.16) has at least three positive roots for $\lambda = \lambda_k$. hence, (2.15) has at least three limit cycles for $\lambda = \lambda_k$.

By replacing $h(r)$ in (2.13) by $h(|r-1|)$, we can show that Theorem 2.2 is no longer valid if the function f is of class C^∞.

From our discussion of (2.15) we know that Theorem 2.3 is a correction of Theorem 10 in [5]. More precisely, the condition that f is analytic should be added to Theorem 10 [5]. Moreover, noting that we do not require that (1.1) is a semicomplete family, Theorem 2.3 is an improvement of Theorem 4 in [9]. We also point out that Duff [5] required that $\theta_\lambda(x, \lambda) \geqslant \varepsilon > 0$ for $0 < |x| \ll 1$ which implies that tr $f_x(0, \lambda) \neq 0$ for some λ. But, for a given (semicomplete) family of rotated vector fields there may hold tr $f_x(0, \lambda) \equiv 0$. an example of semicomplete family with this property is given by

$$\frac{dx}{dt} = y - x^5 + \lambda x^3, \qquad \frac{dy}{dt} = -x.$$

Let for some $\lambda_0 \in I$ (1.1) have a singular closed orbit L_0 such that the Poincaré map is well defined on one side of L_0. Then there is a cross section Σ with an endpoint $A_0 \in L_0$ such that for $A \in \Sigma$ with $A \neq A_0$ either $\alpha(A) = L_0$, $\omega(A) = L_0$ or $\alpha(A)$, $\omega(A) \neq L_0$, where $\alpha(A)$, $\omega(A)$ denote respectively the negative and positive limit sets of the orbit of (1.1) $(\lambda = \lambda_0)$ passing through A. We call L_0 isolated (resp., nonisolated) if $\alpha(A) = L_0$ or $\omega(A) = L_0$ (resp., $\alpha(A)$, $\omega(A) \neq L_0$) for all $A \in \Sigma - \{A_0\}$. Obviously, L_0 is nonisolated if and only if there exists a sequence of periodic orbits approaching L_0.

Then, by using Theorem 2.1 and the Poincaré–Bendixson theorem we can verify easily

THEOREM 2.4. *Suppose* (1.1) *defines a family of rotated vector fields. Let for some* $\lambda_0 \in I$ (1.1) *have a singular closed orbit* $L_0 \subset \mathrm{Cl}\, D$ *such that the Poincaré map is well defined on one side of* L_0. *We have*:

(i) *If* L_0 *is nonisolated,* (1.1) *has no periodic orbits near* L_0 *for* $\lambda \in I$ *near* λ_0 *with* $\lambda \neq \lambda_0$.

(ii) *if* L_0 *is isolated,* (1.1) *has at least one limit cycle near* L_0 *as* λ *is varied in a suitable sense and has no periodic orbits near* L_0 *as* λ *is varied in the opposite sense.*

For Theorem 2.4, we have the following conjecture:

Conjecture. There is at most one limit cycle near L_0 for $\lambda \in I$ satisfying $0 < |\lambda - \lambda_0| \ll 1$ under the condition of Theorem 2.4.

From [6] we have that the conjecture is true in the case of a homoclinic loop if $f(x, \lambda_0) \wedge f_\lambda(x, \lambda_0) \neq 0$ for some $x \in L_0$. If $(x, \lambda_0) \wedge f_\lambda(x, \lambda_0) \equiv 0$ along the homoclinic loop L_0, it is possible to prove the conjecture by using the normal form of displacement function near L_0 obtained by Roussarie [13]. The key point is to prove that the displacement function has a continuous root for $\lambda > \lambda_0$ or $\lambda < \lambda_0$ in a fixed sense.

By Theorem 2.2 it is easy to describe the way the limit cycles of (1.1) terminate. In fact, we have

THEOREM 2.5. *Suppose* (1.1) *defines a family of rotated vector fields for* $(x, \lambda) \in D \times I$ *with* I *an open interval. Let* $G \subset D$ *be a connected region covered by limit cycles of the family* (1.1) *with fixed orientation. Let* L^* *denote its inner or outer boundary. Then* (i) *there exists* $\lambda^* \in \mathrm{Cl}\, I$ *such that* $L(\lambda) \to L^*$ *as* $\lambda \to \lambda^*$ *from one side of* λ^* *where* $L(\lambda)$ *is a limit cycle of* $F(\lambda)$ *in* G; (ii) *if* $\lambda^* \in I$, *then* L^* *is either a critical point, a singular closed orbit, or an unbounded invariant curve.*

Consider the Lienard system

$$\frac{dx}{dt} = y - (x^4 + x^3 - \lambda x), \qquad \frac{dy}{dt} = -x. \tag{2.18}$$

It is easy to see that (2.18) forms a family of rotated vector fields. When $\lambda = 0$ (>0), the origin is a stable (unstable) focus. Hence by Theorem 2.3 for $\lambda > 0$ (2.18) has a unique limit cycle $L(\lambda)$ which expands as λ increases. Set $u(x) = \frac{1}{2}x^4 - \frac{1}{2}\lambda x$. It is direct that for (2.18)

$$\frac{dy}{dx} \geqslant u'(x), \qquad \text{for} \quad y = u(x), \qquad x \leqslant 0,$$

and

$$x^4 + x^3 - \lambda x > u(x) \qquad \text{for} \quad x < 0$$

if $\lambda > \frac{32}{27}$. It follows that (2.18) has an unbounded integral curve $y = U(x)$, $x < 0$ satisfying $U(x) < u(x)$ for $x < 0$ and $U(0-0) = 0$. Hence, there exists $\lambda^* \in (0, \frac{32}{27})$ such that $L(\lambda)$ becomes an unbounded orbit of (2.18) $|_{\lambda=\lambda^*}$ as $\lambda \to \lambda^* - 0$. By Theorem 2.1, (2.18) has no limit cycles for $\lambda \leqslant 0$ or $\lambda \geqslant \lambda^*$.

Finally, we consider the cubic Lienard system

$$\frac{dx}{dt} = y - F(x, \lambda), \qquad \frac{dy}{dt} = -x^3, \tag{2.19}$$

where $F(x, \lambda) = 2x^2(1+x) - \lambda x$, $\lambda \in R$. Let $V = \frac{1}{2}y^2 + \frac{1}{4}x^4$. Then we have

$$\frac{dV}{dt} = x^4[\lambda - 2x^2(1+x)]$$

along orbits of (2.19). Hence, the critical point of (2.19) at the origin is completely unstable (stable) if $\lambda > 0$ ($\lambda < 0$). Note that all the positive orbits of (2.19) are bounded (see [7, 15, 16]). There exists a stable limit cycle, denoted by $L(\lambda)$, of (2.19) for all $\lambda > 0$. $L(\lambda)$ expands monotonically as λ increases and disappears at infinity as $\lambda \to +\infty$. Let us investigate what it becomes as $\lambda \to 0$. Denote by L_0 the limit position of $L(\lambda)$ as $\lambda \to 0^+$. We claim that L_0 is the maximal singular closed orbit of (2.19) ($\lambda = 0$) with the following properties:

(i) The interior of L_0 is the unique nontrivial elliptic sector at the origin.

(ii) All positive orbits outside L_0 have the origin as their positive limit set. Therefore, the Poincaré map is defined neither inside nor outside L_0.

In fact, the above claim follows easily from the fact that

$$F(x, 0) > x^2, \qquad \left.\frac{dy}{dx}\right|_{y=x^2} > 2x = (x^2)' \qquad \text{for} \quad -\frac{1}{4} < x < 0,$$

and

$$F(x, 0) > x^2, \qquad \left.\frac{dy}{dx}\right|_{y=x^2} < 2x \qquad \text{for} \quad 0 < x < +\infty.$$

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